Predicting the Time Dependent Deformation of Viscoelastic Materials Using a Gompertz-type Model

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ABSTRACT

To describe the time dependent deformation of a variety of viscoelastic materials, a one-dimensional nonlinear rheological mathematical model with constant material parameters is developed by using the stress decomposition theory. The use of a logarithmic law as the elastic spring force function required predicts the time versus deformation variation as a Gompertz-type growth function known to be powerful to represent any asymmetric S-shaped experimental data. The predictive quality of the model and its sensitivity to material coefficients are illustrated by numerical examples.

Keywords: Gompertz-type model, logarithmic elastic force function, mathematical modeling, viscoelasticity, Voigt model.

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1. INTRODUCTION

In characterization of materials, the mechanical properties are described as purely elastic or viscoelastic behavior, following that the time dependent effect is neglected or taken into consideration. But, it is well known that real materials are time and history-dependent, to say, viscoelastic materials. Several engineering and biomedical applications using viscoelastic materials require the formulation of time dependent deformation model. There are, in viscoelastic modeling, two categories of theory. The first is the classical linear viscoelastic theory, which is represented usually in the Boltzmann single integral form or in differential equation. This well known established theory is, however, only valid for small deformations or low stresses. The second is the nonlinear viscoelastic theory which has not, contrary to the linear theory, a definitive framework formulation. Since viscoelastic materials exhibit time dependent highly large deformations, the linear viscoelastic theory is inapplicable and then, nonlinear viscoelastic models are required. For example, it is well known in biomechanical studies that arterial tissue undergoes large deformations when it is subjected to physiological load. Thus its mechanical properties are essentially nonlinear and could not be represented on the basis of the classical linear viscoelasticity. Different theoretical formulations of varying complexities have been developed for investigating the nonlinear time dependent properties of viscoelastic materials (Chotard-Ghodsnia and Verdier, 2007). However, these models fail to include the inertia of the mechanical system studied in the constitutive equations, in the perspective that viscoelastic materials are characterized simultaneously not just by elastic and viscous contributions, but also by an inertial function. Moreover, there are only a few theoretical models that are formulated with constant-value material coefficients so that, the material functions are considered as stress, strain or strain rate dependent. Due to material nonlinearities, a consistent constitutive equation should take then into account all together the elastic, viscous and inertial nonlinearities and relate mathematically stress, strain and their higher time derivatives (Bauer et al., 1979; Bauer, 1984). In mechanics, the use of rheological models consisting of a combination of spring and dashpot is proved useful to describe viscoelastic behavior of materials. These rheological models are interesting, since they represent the dynamic response of materials concerned in terms of differential equations that can be solved for various particular cases of consideration. So much constitutive equations are derived from these combinations of spring and dashpot in order to predict and simulate material properties, and analyze experimental data. In this regard, to model materials nonlinear properties, the linear viscoelastic theory can be modified and extended to higher order stress or strain terms. A number of recent successful theoretical models have been developed on the basis of classical linear viscoelastic models extension to large deformations (Monsia, 2011a, 2011b, 2011c, 2011d). In Monsia (2011a, 2011b, 2011c), only the elastic nonlinearity is taken into account by introducing a nonlinear spring force in the classical linear rheological models. Consequently, these models are insufficient to account for complete characterization of viscoelastic materials. The model (Monsia, 2011d), which is a nonlinear generalized Maxwell fluid model, taking into consideration both elastic and viscous nonlinearities, appeared useful for representing accurately the viscoelastic materials time-dependent behavior. In (Monsia, 2011d), however, the inertial contribution is neglected. In contrast to these models, the model (Monsia, 2011e) has been constructed by taking into consideration the elastic, viscous and inertial nonlinearities simultaneously. The model (Monsia, 2011e) attempted successfully to represent mathematically a complete characterization of viscoelastic materials. This model (Monsia, 2011e) was founded on the stress decomposition theory developed previously by Bauer (1984) for a complete characterization of viscoelastic arterial walls. The Bauer's theory (1984) allows, in effect, solving the mathematical complexities in rheological modeling and accounting simultaneously for high elastic, viscous and inertial nonlinearities characterizing viscoelastic materials. The Bauer's theory (1984) is derived from the classical Voigt model. In this theory

(Bauer, 1984), the total stress acting on the material is decomposed as the sum of three components, that is, the elastic, viscous and inertial stresses. The purely elastic stress is written as a power series of strain, the purely viscous stress as a first time derivative of a similar power series of strain, and the purely inertial stress as a second time derivative of a similar power series of strain. The Bauer's stress decomposition method (1984) has been after used by many authors (Armentano et al., 1995; Gamero et al., 2001; Monsia et al., 2009) for the complete characterization of arterial behavior. In (Monsia et al., 2009), following the Bauer's approach (1984), the elastic stress is expanded in power series of strain. Monsia (2011e), using the Bauer's method (1984), developed a hyperlogistic equation that represents successfully the time-dependent mechanical properties of viscoelastic materials by expressing the elastic stress as an asymptotic expansions in powers of deformation, the viscous stress as a first time derivative of a similar asymptotic expansions in powers of deformation, and the inertial stress as a second time derivative of a similar asymptotic expansions in powers of deformation. Recently, Monsia (2011f, 2012) formulated in a single differential equation the Bauer's stress decomposition theory (1984) with an exciting stress term, depending on a nonlinear elastic spring force function $\varphi(\varepsilon)$, where the scalar function $\varepsilon(t)$ represents the time dependent deformation of the mechanical system under study. In (Monsia, 2011f), the function $\varphi(\varepsilon)$ is written as a hyperbolic function, which led, in the absence of exciting stress, the author to obtain, after an adequate mathematical manipulation, a useful hyper-exponential type function representing the time versus strain variation of the viscoelastic material considered. The same author (Monsia, 2012), considering also the hyperbolic elastic spring force law $\varphi(\varepsilon)$, with now the presence of a constant exciting stress, developed successfully, after consistent mathematical operations, a nonlinear mechanical model applicable for representing the nonlinear creep behavior of viscoelastic materials. More recently, in Monsia and Kpomahou (2012), the authors, by using the Bauer's theory as formulated previously in Monsia (2011f, 2012), and expressing the nonlinear elastic spring force function $\varphi(\varepsilon)$ in a Newton's binomial function, constructed successfully a four-parameter mechanical model to represent the dynamic response of viscoelastic materials. In Monsia and Kpomahou (2012), the binomial law exponent controlled the material model nonlinearity. Numerical applications performed by the authors (Monsia and Kpomahou, 2012), clearly showed the powerful predictive ability of the model to reproduce any S-shaped experimental data. These studies demonstrate the authoritative suitability of the Bauer's stress decomposition theory (1984) as an advanced mathematical tool in rheological modeling. The use of the Bauer's theory (1984) requires overcoming two major difficulties. The first consists of a suitable choice of the nonlinear elastic force function $\varphi(\varepsilon)$ that should tend towards the expected linear hookean behavior for small deformations. The second difficulty results in the fact that the application of the Bauer's theory (1984) leads often to solve a Liénard second order nonlinear ordinary differential equation that is generally non-integrable. These considerations show that the use of the Bauer's theory (1984) to model the material nonlinear time dependent properties is not a simple task. In this paper, considering also the Monsia formulation (2011f, 2012) of the Bauer's approach (1984), a one-dimensional nonlinear rheological model with constant material parameters that includes elastic, viscous and inertial nonlinearities simultaneously, is developed, by using a logarithmic elastic spring force law. The obtained results predicted the time dependent deformation of the viscoelastic material studied as a Gompertz-type function. The model appeared then useful to predict mathematically the time versus deformation variation of viscoelastic materials exhibiting an asymmetric sigmoid mechanical behavior. Numerical illustrations are performed to demonstrate the predictive capability of the model and its sensitivity to material parameters.

2. FORMULATION OF THE MECHANICAL MODEL

2.1 Theoretical Considerations

The present part is devoted to describe the governing equations of the theoretical model including the nonlinear elastic, viscous and inertial contributions characterizing viscoelastic materials. As pointed out previously in Monsia (2011f, 2012), the nonlinear ordinary differential equation resulting from the use of the Bauer's theory (1984), by superposing the pure elastic, viscous and inertial stresses, for a nonlinear elastic spring force function $\varphi(\varepsilon)$, can be written in the form

$$\ddot{\varepsilon}\frac{d\varphi}{d\varepsilon}+\dot{\varepsilon}^2\frac{d^2\varphi}{d\varepsilon^2}+\frac{b}{c}\dot{\varepsilon}\frac{d\varphi}{d\varepsilon}+\frac{a}{c}\varphi(\varepsilon)=\frac{1}{c}\sigma_t \tag{1}$$

The dot over a symbol denotes a differentiation with respect to time t. The inertial module c is different from zero and time independent. The parameters a and b are respectively the stiffness and viscosity coefficients. They are also time independent material parameters. σ_t, which is a scalar function, means the total exciting stress acting on the mechanical system studied. It is required, in order to progress in the present modeling, to identify the nonlinear elastic force function $\varphi(\varepsilon)$ of interest. As stated earlier, the function $\varphi(\varepsilon)$ should obey to the basic principle governing the Bauer's theory, that is to say, behave linearly as the classical hookean elastic spring force function, for small values of deformation $\varepsilon(t)$. Following this principle, in the present study, the nonlinear elastic force function $\varphi(\varepsilon)$ is empirically expressed in terms of a logarithmic function given below

$$\varphi(\varepsilon)=\ln(\varepsilon_o-\frac{\varepsilon}{\varepsilon_o}) \tag{2}$$

where $\varepsilon_o \neq 0$, is a material constant, and $\ln$ denotes the natural logarithm. By using Equation (2), Equation (1) becomes

$$\sigma_t=-c\frac{\ddot{\varepsilon}(\varepsilon_o^2-\varepsilon)+\dot{\varepsilon}^2}{(\varepsilon_o^2-\varepsilon)^2}-b\frac{\dot{\varepsilon}}{\varepsilon_o^2-\varepsilon}+a\ln(\varepsilon_o-\frac{\varepsilon}{\varepsilon_o}) \tag{3}$$

Equation (3) shows mathematically in the single differential form the constitutive relation between the total exciting stress σ_t and the resulting strain $\varepsilon(t)$. Equation (3) represents a second order nonlinear ordinary differential equation in $\varepsilon(t)$ for a given exciting stress σ_t.

2.2 Dimensionalization

The strain $\varepsilon(t)$ is a dimensionless quantity. Then in Equation (3) the coefficients have the following dimensions. Let M, L and T denote the mass, length and time dimension respectively, the dimension of the stress varies as $ML^{-1}T^{-2}$. Therefore, the dimension of a varies as $ML^{-1}T^{-2}$, that of b varies as $ML^{-1}T^{-1}$, and that of c varies as ML^{-1} (mass per unit length).

2.3 Solution using an exciting stress $\sigma_t=0$

<u>2.3.1 Evolution Equation of Deformation</u> $\varepsilon(t)$

In the absence of exciting stress ($\sigma_t=0$), the internal dynamics of the mechanical system under study is governed by the following nonlinear ordinary differential equation

$$c\ddot{\varepsilon}(\varepsilon_o^2-\varepsilon)+c\dot{\varepsilon}^2+b\dot{\varepsilon}(\varepsilon_o^2-\varepsilon)-a(\varepsilon_o^2-\varepsilon)^2\ln\left(\frac{\varepsilon_o^2-\varepsilon}{\varepsilon_o}\right)=0 \tag{4}$$

Equation (4) represents analytically the nonlinear evolution equation of deformation $\varepsilon(t)$ of the considered mechanical system.

2.3.2 Solving Time-Deformation Equation

For solving Equation (4), a change of variable is needed. Making the following suitable substitution

$$\exp(x)=\frac{\varepsilon_o^2-\varepsilon}{\varepsilon_o} \tag{5}$$

Equation (4) transforms, after a few algebraic operations, in the form

$$\ddot{x}+\lambda\dot{x}+\omega_o^2 x=0 \tag{6}$$

where

$\lambda=\dfrac{b}{c}$, and $\omega_o^2=\dfrac{a}{c}$.

Equation (6) is the well-known second-order linear ordinary differential equation which describes a damped harmonic oscillator motion. The solution of Equation (6) depends on the relative magnitudes of λ^2 and ω_o^2, that determine whether the roots of characteristic equation associated with Equation (6) are real or complex numbers. Therefore, three particular cases may be studied.

2.3.2.1 Case A: $\lambda>2\omega_o$

If the damping is relatively large, that is to say, $\lambda>2\omega_o$, the roots of the characteristic equation are real quantities, and the oscillator is said to be overdamped. Thus, the mechanical system dissipates the energy by the damping force and the motion will not be oscillatory. The amplitude of the vibration will decay exponentially with time. In this particular case, integration of Equation (6) yields for $x(t)$ the following solution

$$x(t)=A_1\exp(r_1 t)+A_2\exp(r_2 t) \tag{7}$$

where

$$r_1=-\frac{\lambda}{2}(1+\delta)$$

and

$$r_2=-\frac{\lambda}{2}(1-\delta)$$

are the two negative real roots of the characteristic equation

$$r^2+\lambda r+\omega_o^2=0$$

with

$$\delta=\sqrt{1-4\frac{\omega_o^2}{\lambda^2}}$$

A_1 and A_2 are two integration constants determined by the initial conditions. Thus, using the suitable initial conditions

$t=0$, $\varepsilon(t)=0$
and
$t=0$, $\dot{\varepsilon}(t)=0$
and taking into consideration Equation (5), one can obtain the following explicit analytical solution

$$\varepsilon(t)=\varepsilon_o^2-\varepsilon_o\exp\left[\ln(\varepsilon_o)\left(\frac{1+\delta}{2\delta}\exp(-\frac{\lambda}{2}(1-\delta)t)-\frac{1-\delta}{2\delta}\exp(-\frac{\lambda}{2}(1+\delta)t)\right)\right] \tag{8}$$

Equation (8) gives the strain versus time relationship of the viscoelastic material under study. It predicts mathematically the time dependent deformation response of the material studied as a Gompertz-type model that is useful for representing an asymmetric sigmoid curve.

2.3.2.2 Case B: $\lambda=2\omega_o$

For $\lambda=2\omega_o$, the oscillator is said to be critically damped and the amplitude of the vibration will decay without sinusoidal oscillations during the time. In this case, Equation (6) has the solution of the form

$$x(t)=(B_1t+B_2)\exp(-\frac{\lambda}{2}t) \tag{9}$$

where B_1 and B_2 are two integration constants determined by the initial conditions. Therefore, using the suitable initial conditions
$t=0$, $\varepsilon(t)=0$
and
$t=0$, $\dot{\varepsilon}(t)=0$
and considering also Equation (5), the desired solution $\varepsilon(t)$ may be written in the following form

$$\varepsilon(t)=\varepsilon_o^2-\varepsilon_o\exp(\ln(\varepsilon_o)(1+\frac{\lambda}{2}t)\exp(-\frac{\lambda}{2}t)) \tag{10}$$

Equation (10) describes also the strain time relationship as a Gompertz-type function adequate to fit the asymmetric S-shaped experimental data.

2.3.2.3 Case C: $\lambda<2\omega_o$

For a relatively small damping, to say, $\lambda<2\omega_o$, the roots of the characteristic equation are complex numbers, and the oscillator is said to be underdamped. The amplitude of the vibration decreases exponentially with time. In this particular case, integration of Equation (6) yields for $x(t)$ the following solution

$$x(t)=C\exp(-\frac{\lambda}{2}t)\cos(\omega t-\phi) \tag{11}$$

where

$$\omega=\sqrt{\omega_o^2-\frac{\lambda^2}{4}}$$

and C and ϕ are two integration constants determined by the initial conditions. Then, setting the suitable initial conditions
$t=0$, $\varepsilon(t)=0$
and

$t=0$, $\dot{\varepsilon}(t)=0$

and taking into consideration Equation (5), the following explicit analytical solution for the desired strain $\varepsilon(t)$ can be obtained

$$\varepsilon(t)=\varepsilon_o^2-\varepsilon_o \exp(\ln(\varepsilon_o)(\cos(\omega t)+\frac{\lambda}{2\omega}\sin(\omega t))\exp(-\frac{\lambda}{2}t)) \quad (12)$$

The exponentiated exponential Equation (12) is of the form of a Gompertz-type model in which the constant parameter $\ln(\varepsilon_o)$ is modulated by the sinusoidal function $\cos(\omega t)+\frac{\lambda}{2\omega}\sin(\omega t)$, and appears very useful for the asymmetric S-shaped experimental data fitting.

3. NUMERICAL RESULTS AND DISCUSSION

This section presents some numerical examples to investigate the predictive capability of the model to reproduce the mechanical response of the material considered. The dependence of strain versus time curve on the material parameters is also discussed.

Figure 1 illustrates the typical time dependent strain behavior of viscoelastic materials studied, resulting from Equation (8) with the fixed value of coefficients at $\lambda=2$, $\omega_o=0.5$, $\varepsilon_o=6$. We note that the strain $\varepsilon(t)$ increases until the asymptotical value $\varepsilon_o^2-\varepsilon_o$ with increase time t, and the elastic spring force $\varphi(\varepsilon)$ becomes then equal to zero. Therefore, the slope, to say, the strain rate, after reaching its peak value at the inflexion point, declines gradually with time t until the failure point at which it reduces to zero. It is easy to observe from Figure 1 that the model can predict accurately the typical sigmoid deformation of some viscoelastic materials, to say, for example, soft living tissues (Monsia, 2011f, 2012; Monsia and Kpomahou, 2012). The strain versus time curve is nonlinear, with a nonlinear beginning initial portion. The plotting illustrates then the S-shaped deformation behavior of the viscoelastic material studied.

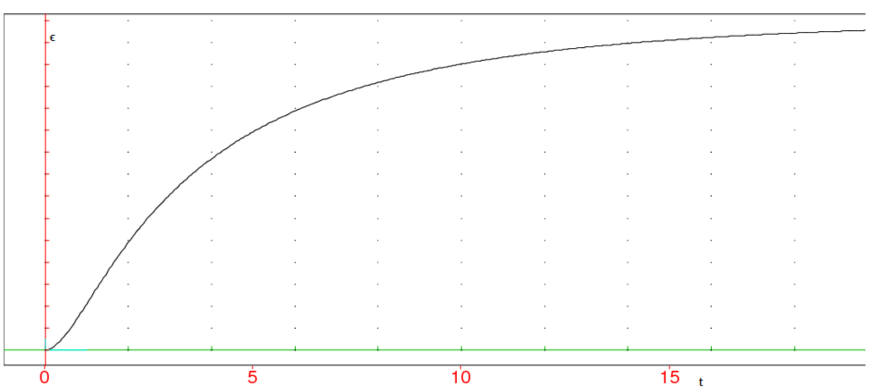

Fig.1. Typical strain versus time curve exhibiting a sigmoid behavior.

Figure 2, 3 and 4, illustrates the effects of material coefficients on the strain versus time curve generated by Equation (8). The effects of the action of these coefficients are studied with the help of an own computer program by varying step by step one coefficient while the other two are kept constant. As shown in Figure 2, an increase of the viscosity coefficient λ, has no significant effect on the initial value of the strain, but decreases the value of the strain on the time period considered. The slope also decreases with increase λ. The red line corresponds to $\lambda = 2$, the blue line to $\lambda = 3$, and the green line to $\lambda = 4$. The other parameters are $\omega_o = 0.5$, $\varepsilon_o = 6$.

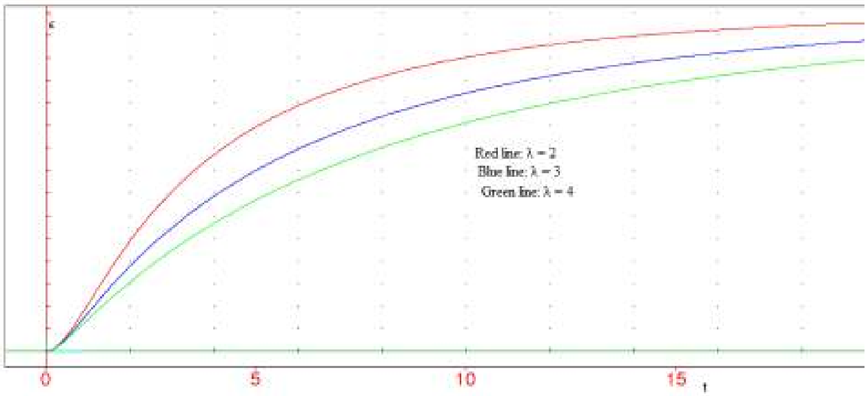

Fig. 2. Strain versus time curve at various values of the viscosity coefficient λ.

Figure 3 shows the effect of the natural frequency ω_o variation on the strain-time response. An increase ω_o, increases the strain value on the time period considered, increases also the slope and the curves become more nonlinear. However, an increasing ω_o, has no important effect on the initial value of the strain. The red line corresponds to $\omega_o = 0.05$, the blue line to $\omega_o = 0.1$, and the green line to $\omega_o = 0.5$. The other parameters are $\lambda = 2$, $\varepsilon_o = 6$.

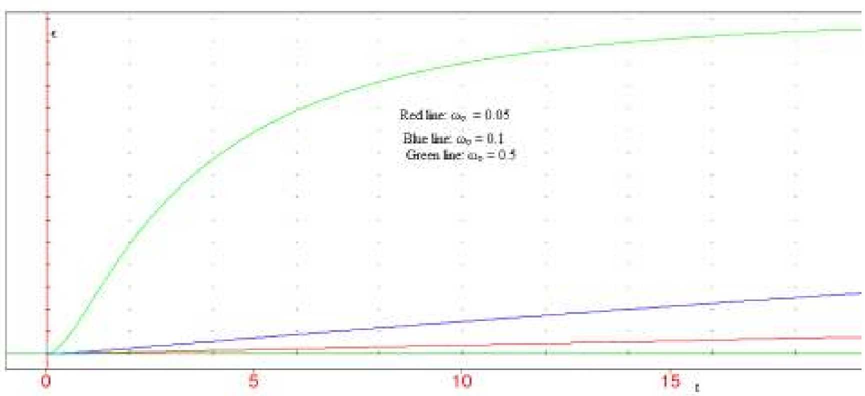

Fig. 3. Strain-time curves with three different values of the natural frequency ω_o.

From Figure 4, we note that change of the coefficient ε_o has a high effect on the peak asymptotical value of the strain. We observe that an increase ε_o, increases significantly and fast the maximum asymptotical value of the strain. The slope increases also with increase ε_o. But, an increase ε_o, has no significant effect on the initial value of the strain. The red line corresponds to $\varepsilon_o = 6$, the blue line to $\varepsilon_o = 8$, and the green line to $\varepsilon_o = 10$. The other parameters are $\lambda = 2$, $\omega_o = 0.5$.

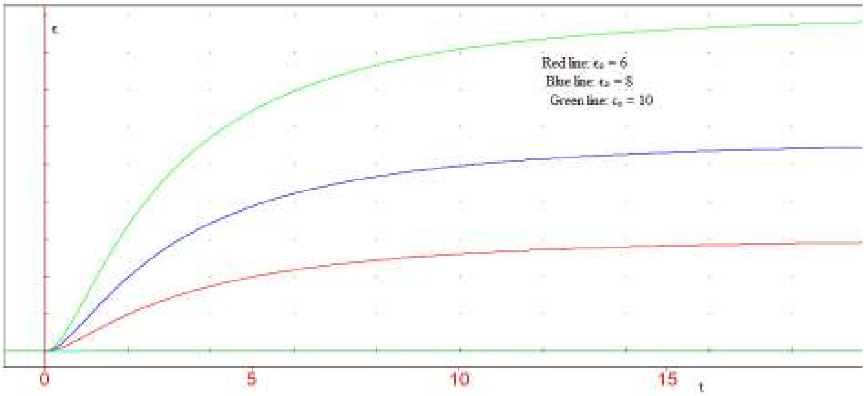

Fig. 4. Strain versus time curves showing the effect of the coefficient ε_o.

Figures 5 and 6 show the typical strain versus time curves derived from Equation (10) and Equation (12), with the fixed value of coefficients at $\lambda = 1$, $\varepsilon_o = 2$, and $\lambda = 1$, $\omega_o = 1$, $\varepsilon_o = 2$, respectively. These curves exhibit the same limiting value of $\varepsilon(t) = \varepsilon_o^2 - \varepsilon_o$. Thus, as time t increases, the strain rate, that is to say, the slope of these curves, after reaching its peak value at the inflexion point, decreases progressively until the failure point at which it reduces to zero. The curves illustrate the nonlinear sigmoid behavior of materials of interest.

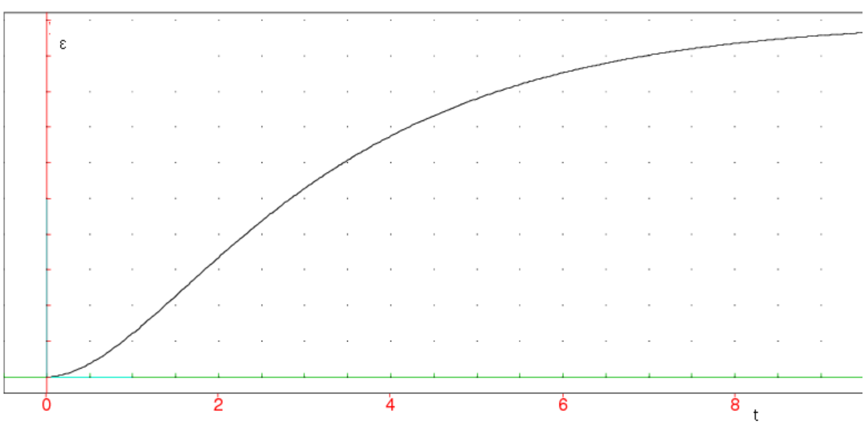

Fig.5. Typical strain time curve showing the S-shaped behavior derived from Equation (10).

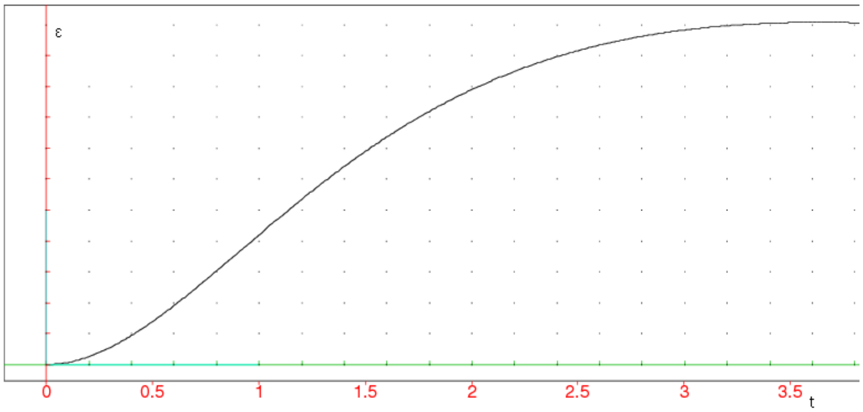

Fig.6. Typical strain versus time curve exhibiting the S-shaped behavior generated by Equation (12).

The preceding numerical examples demonstrated that the model is well-suited to represent the asymmetric S-shaped deformation response of viscoelatic materials. The model is based on the Bauer's theory (1984) consisting to superpose the elastic, viscous and inertial nonlinear contributions for obtaining the total stress acting on the material. This method permitted to perform a complete characterization of the viscoelastic material under study. In this model, the nonlinear elastic force function is assumed to be a logarithmic law, which allowed taking into account elastic, viscous and inertial nonlinearities simultaneously, and deriving successfully the time dependent response of the material studied as a Gompertz-type function that is well known useful for reproducing an asymmetric sigmoid curve. It is also interesting to note that the Gompertz model is an asymmetric function widely used to represent increases in several growth phenomena exhibiting a sigmoid pattern, for example, in physics, biology and biomedical science. The empirical choice of the nonlinear logarithmic elastic force function $\varphi(\varepsilon)$ is inspired by the work (Covács et al., 2001) and also justified by the fact that for $\varepsilon << \varepsilon_o$, the function $\varphi(\varepsilon)$ can be developed in power series of deformation ε. In this regard, the choice of function $\varphi(\varepsilon)$ agrees very well with the polynomial function of deformation utilized by Bauer (1984) so that, for small values of deformation, $\varphi(\varepsilon)$ behaves linearly as expected.

4. CONCLUSION

To describe the nonlinear time dependent properties of materials, a mechanical model has been developed by using the stress decomposition theory. The nonlinear elastic, viscous and inertial contributions characterizing viscoelastic materials are simultaneously taking into consideration through the use of a logarithm law for the nonlinear elastic spring force function in the present model. The model appeared useful to predict the dynamic deformation behavior of materials as a Gompertz-type sigmoid function.

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